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BOOK: Differential Geometry and Tensor-1

UNIT –I : Space curves, Tangent, Contact of Curve and Surfaces, Osculating Plane

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OBJECTIVES

After studying this unit, you will be able to know about the-

- **Differential Geometry.**
- **Space Curves.**
- **Tangents.**
- **Contact of Curve and Surfaces.**
- **Osculating Plane.**

INTRODUCTION

Differential geometry is that part of geometry which is treated with the help of differential calculus. There are two branches of differential geometry:

Local differential geometry: In which we study the properties of curves and surfaces in the neighborhood of a point.

Global differential geometry: In which we study the properties of curves and surfaces as a whole.

SPACE CURVE

A curve in space is defined as the locus of a point whose cartesian coordinates are the functions of a single variable parameter u, say.

We can represent a space curve in the following two ways :

As intersection of two surfaces :

Let $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$ be two surfaces then these equations together represent the curve of intersection of the above surfaces. If this curve lies in a plane then it is called a plane curve, otherwise it is called to be skew, twisted or tortous.

For example, if $f_1(x, y, z) = 0$, represents a sphere and $f_2(x, y, z) = 0$ represents a plane then these two equations together represent a circle.

Parametric representation :

If the coordinates of a point on a space curve be represented by the equations of the following form

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t) \qquad \text{.....(1.2.1)}$$

where f_1, f_2, f_3 are real valued functions of a single real variable t ranging over a set of values $a \le t \le b$.

The equation in (1.2.1) are called parametric equation of the space curve.

VECTOR REPRESENTATION OF SPACE CURVE

If $\vec{r}$ be the position vector of a current point A on the space curve whose cartesian coordianates be x, y, z then we know that

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

or $$\vec{r} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

or $$\vec{r} = f(t)$$

or $$\vec{r} = (f_1(t), f_2(t), f_3(t)) \qquad .....(1.2.2)$$

where f is a vector valued function of a single variable t. Thus space curve may be defined as :

A space curve is the locus of a point whose position vector $\vec{r}$ with respect to a fixed origin may be expressed as a function of single parameter.

UNIT TANGENT VECTOR OF A CURVE

Consider two neighbouring points A (x, y, z) and B $(x + \delta x, y + \delta y\ z + \delta z)$ on a curve C whose position vectors are r and $r + \delta r$, respectively. We have

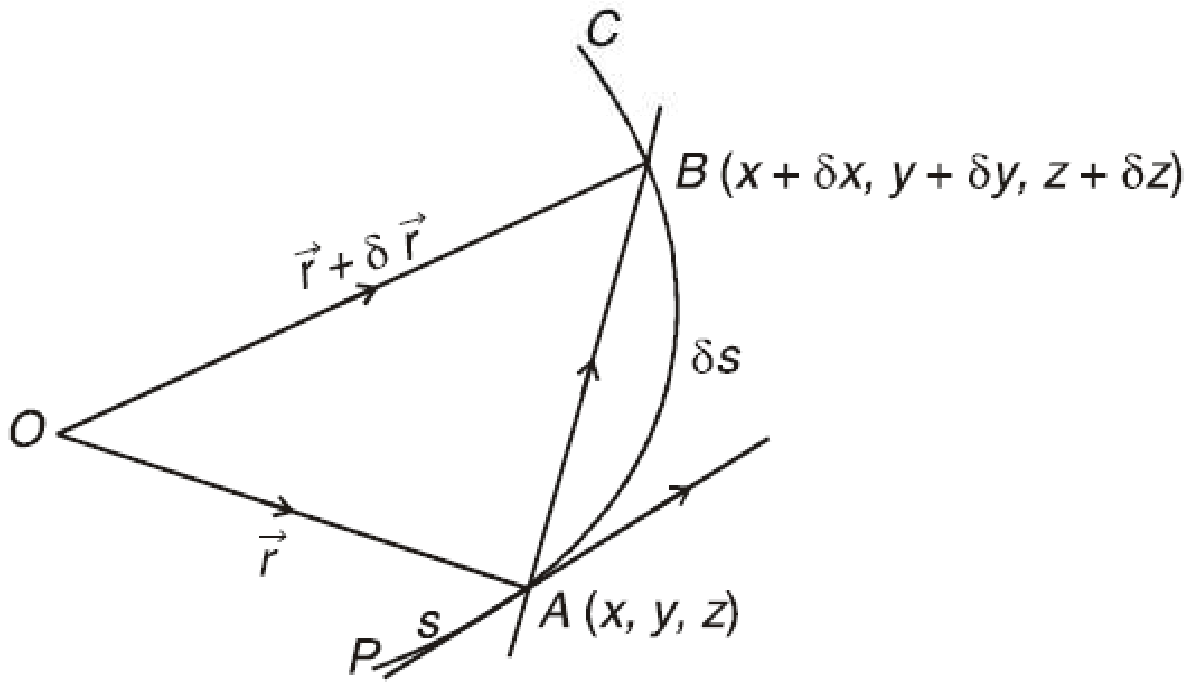

Fig. 1.1

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{r} + \delta\vec{r} - \vec{r} = \delta\vec{r}\,.$$

Let δs be length of arc AB measured along the curve and arc $PA = s$ is measured from any convenient point P on the curve.

$$\text{Unit vector along chord } AB = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{\delta\vec{r}}{\text{Chord } AB}$$

$$= \frac{\delta\vec{r}}{\delta s} \cdot \frac{\text{Arc } AB}{\text{Chord } AB} \qquad .....(1.2.3)$$

But as B tends to A, then the chord AB tends to be tangent at P.

Also we know that $$\lim_{B \to A} \frac{\text{Arc } AB}{\text{Chord } AB} = 1$$

Hence, unit vector along tangent at A $= \lim_{B \to A} \dfrac{\delta \vec{r}}{\delta s} \cdot \dfrac{\text{Arc } AB}{\text{Chord } AB} = \dfrac{d\,\vec{r}}{ds} \cdot 1$

$$= \frac{d\,\vec{r}}{ds} = \vec{r}\,' \qquad \text{.....(1.2.4)}$$

Unit tangent vector at A is denoted by $\hat{t}$ and is taken in the direction of s increasing

If $\quad \vec{r} = (x, y, z)$ *i.e.* $\vec{r} = x\hat{i} + y\hat{i} + z\hat{k}$

then $\quad \hat{t} = \dfrac{d\vec{r}}{ds} = \left(\dfrac{dx}{ds}, \dfrac{dy}{ds}, \dfrac{dz}{ds} \right)$

i.e.
$$\hat{t} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k} \qquad \text{.....(1.2.5)}$$

Since $\hat{t}$ is unit tangent vector, $\left| \hat{t} \right| = 1$.

$\therefore \qquad 1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2$

or $\qquad 1 = \left(\frac{dx}{dt}\cdot\frac{dt}{ds}\right)^2 + \left(\frac{dy}{dt}\cdot\frac{dt}{ds}\right)^2 + \left(\frac{dz}{dt}\cdot\frac{dt}{ds}\right)^2$

or $\qquad \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$

or $\qquad \dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2,$

where $\qquad \dot{s} = \frac{ds}{dt}, \dot{x} = \frac{dx}{dt}$ *etc.* $\qquad$ .....(1.2.6)

and t is any parameter.

THE EQUATION OF TANGENT LINE TO A CURVE AT A GIVEN POINT

The tangent line to a curve at any point A is defined as the limiting position of a straight line through the point A and a neighbouring point B on the curve as B tends to A along the curve.

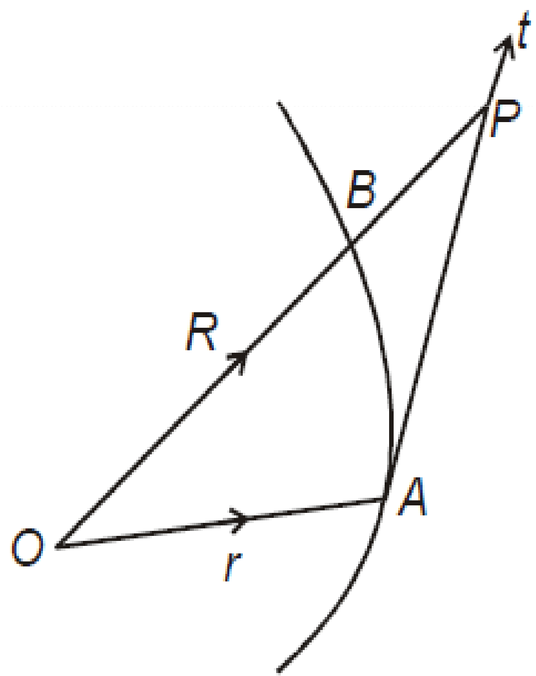

Fig. 1.2

Let $\vec{r} = \vec{r}(s)$ be the parametric equation of a curve and A be any point on it whose position vector is $\vec{r}$ and a unit tangent vector at A be denoted by $\hat{t} = \frac{d\vec{r}}{ds} = \vec{r}'$.

Let P be any point on the tangent line at A whose position vector is $\vec{R}$ (say).

Also $\overline{AP} = w\,\hat{t}$ where $\left|\overline{AP}\right| = w$

But $\overline{OP} = \overline{OA} + \overline{AP}$

$\therefore$ $\vec{R} = \vec{r} + w\,\hat{t}$ or $\vec{R} = \vec{r} + w\,\vec{r}'$ .....(1.2.7)

Equation (1.2.2) gives us the equation of tangent line at A.

Tangent line in cartesian form :

We may write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow$ $\vec{r}' = x'\hat{i} + y'\,\hat{j} + z'\hat{k}$

and $$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

Substituting these value in equation (1.2.2) of tangent line, we get

$$X\hat{i} + Y\hat{j} + Z\hat{k} = x\hat{i} + y\hat{j} + z\hat{k} + c\left(x'\hat{i} + y'\hat{j} + z'\hat{k}\right)$$

or $$X\hat{i} + Y\hat{j} + Z\hat{k} = \left(x + cx'\right)\hat{i} + \left(y + cy'\right)\hat{j} + \left(z + cz'\right)\hat{k},$$

where c is a non-zero constant.

Equating coefficients of $\hat{i}, \hat{j}, \hat{k}$ from both sides

$$X = x + cx',\ Y = y + cy',\ Z = z + cz'$$

i.e. $$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'} = c,$$

i.e. $$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'} \qquad .....(1.2.8)$$

This is the required equation of tangent line at (x, y, z) and direction cosines of the tangent line are proportional to x', y', z'.

EQUATION OF TANGENT LINE WHEN THE EQUATION OF THE GIVEN CURVE IS GIVEN AS THE INTERSECTION OF TWO SURFACES

Let the equation of two surfaces are

$$F_1(x, y, z) = 0 \text{ and } F_2(x, y, z) = 0 \qquad .....(1.2.9)$$

where x, y, z are functions of a parameter.

Now $$\frac{\partial F_1}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial F_1}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial F_1}{\partial z}\cdot\frac{dz}{dt}=0 \qquad .....(1.2.10)$$

$$\frac{\partial F_2}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial F_2}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial F_2}{\partial z}\cdot\frac{dz}{dt}=0 \qquad .....(1.2.11)$$

Hence from equation (1.2.3) and (1.2.4)

$$\frac{\dot{x}}{\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z}-\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}}=\frac{\dot{y}}{\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}}=\frac{\dot{z}}{\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y}-\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}}$$

which are the direction ratios of the tangent and dot represents differentiation w.r. to '*t*'.

Therefore, the equation of tangent line at a point (x, y, z) on the curve of intersection of the two given surfaces is given as

$$\frac{X-x}{\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z}-\frac{\partial F_1}{\partial z}.\frac{\partial F_2}{\partial y}}=\frac{Y-y}{\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}.\frac{\partial F_2}{\partial z}}=\frac{Z-z}{\frac{\partial F_1}{\partial x}.\frac{\partial F_2}{\partial y}-\frac{\partial F_1}{\partial y}.\frac{\partial F_2}{\partial x}} \quad .....(1.2.13)$$

DIRECTION-COSINES OF THE TANGENT LINE

Let $A(x, y, z)$ and $B(x+\delta x, y+\delta y, z+\delta z)$ be adjacent points on a given curve in rectangular coordinate axes. δr the measure of chord AB is given by

$$\delta\bar{r}^2 = \delta x^2 + \delta y^2 + \delta z^2$$

Let s be the length of the arc measure from some fixed point P to any point A on the curve.

If the measure of the arc AB of the curve be δs then

$$\left(\frac{\delta r}{\delta s}\right)^2 = \left(\frac{\delta x}{\delta s}\right)^2 + \left(\frac{\delta y}{\delta s}\right)^2 + \left(\frac{\delta z}{\delta s}\right)^2$$

Since $$\lim_{B\to A} \frac{\text{Chord } AB}{\text{Arc } AB} = 1$$

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2,$$

or $$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = |\dot{r}|^2$$

Hence $$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \qquad .....(1.2.14)$$

where x, y, z are functions of t and $\dot{x} = \frac{dx}{dt}$ etc.

But $\dot{x}, \dot{y}, \dot{z}$ are direction ratios of a tangent line therefore the direction cosines of the tangent line at A are

$$\frac{\dot{x}}{\dot{s}}, \frac{\dot{y}}{\dot{s}}, \frac{\dot{z}}{\dot{s}} \quad \text{or} \quad \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$$

But $$\frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}.$$

The direction cosines of the tangent line are x', y', z' which are the components of r' where a prime denotes differentiation with respect to s. Clearly $|\vec{r}'| = 1$, *i.e.* $\vec{r}'$ is unit vector along the tangent.

Examples:

Ex.1. *Find the equation to the tangent at the point* θ *on the circular helix*

$$x = a\cos\theta,\ y = a\sin\theta,\ z = C\theta$$

Sol. The vector equation of the helix is given by

$$\vec{r} = a\cos\theta\hat{i} + a\sin\theta\hat{j} + C\,\theta\,\hat{k}$$

$$\vec{r}' = -\,a\sin\theta\hat{i} + a\cos\theta\hat{j} + C\,\hat{k}$$

The equation of the tangent in given by

$$\vec{R} = \vec{r} + \lambda\vec{r}'$$

or
$$\vec{R} = \left(a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + C\,\theta\,\hat{k}\right) + \lambda\left(-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + C\,\hat{k}\right)$$

If
$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k},$$

then
$$X\hat{i} + Y\hat{j} + Z\hat{k} = a(\cos\theta - \lambda\sin\theta)\,\hat{i} + a(\sin\theta + \lambda\cos\theta)\,\hat{j} + C(\theta + \lambda)\,\hat{k}$$

which gives
$$\frac{X - a\cos\theta}{-a\sin\theta} = \frac{Y - a\sin\theta}{a\cos\theta} = \frac{Z - C\theta}{C}.$$

It is the required equation of tangent line.

Ex. **2.** *Show that the tangent at any point of the curve whose equations are*

$$x = 3t,\ y = 3t^2,\ z = 2t^3$$

makes a constant angle with line

$$y = z - x = 0.$$

Sol. The direction-rations of the tangent at 't' to the given curve are

$$3,\ 6t,\ 6t^2 \quad (i.e.,\ \dot{x}, \dot{y}, \dot{z})$$

The direction ratios of the given line are

$$1,\ 0,\ 1.$$

If θ be the angle between the tangent and the given line, than

$$\cos\theta = \frac{3\times 1 + 6t\times 0 + 6t^2\times 1}{\left(\sqrt{9+36t^2+36t^4}\right)\left(\sqrt{1+0+1}\right)}$$

$$= \frac{3\left(1+2t^2\right)}{\sqrt{2}\times 3\left(1+2t^2\right)} = \frac{1}{\sqrt{2}}$$

which is independent of t, hence θ is constant.

CONTACT OF CURVE AND SURFACES

We know that in a plane curve the tangent at A is the limiting position of the chord AB when B coincides with A. In a similar manner if $A_1, A_2, \ldots, A_{n+1}$ be points on a given curve lying on a given surface and if $A_2, A_3, \ldots, A_{n+1}$ all coincide with A_1, than we say that a curve has a contact of nth order with the surface at A_1. We may also say that the curve and the surface has $(n + 1)$ points of contact.

1.3.1 Definition :

If $A, A_1, A_2, \ldots, A_n$ points on a given curve lie on a given surface and $A_1, A_2, \ldots, A_n$ coincide with A, then curve and surface are said to have the contact of nth order at the point A.

CONDITION FOR A CURVE AND A SURFACE HAVE A CONTACT OF n^{th} ORDER

Let the equation of the curve C be given by

$$\vec{r} = \{x(t), y(t), z(t),\} \qquad .....(1.3.1)$$

and the equation of the surface S be given by

$$f(x, y, z) = 0 \qquad .....(1.3.2)$$

The values of t corresponding to the points of intersection of the curve C and surface S are the roots of the equation

$$F(t) = f\{x(t), y(t), z(t)\} = 0 \qquad .....(1.3.3)$$

Let $t = t_0$ be a root of the equation $F(t) = 0$ so that

$$F(t_0) = 0. \qquad .....(1.3.4)$$

Then $t = t_0$ give as a point of intersection of C and S.

Put $t = t_0 + h$ so that

$$F(t) = F(t_0 + h). \qquad .....(1.3.5)$$

Expanding $F(t)$ about t_0 by Taylor's theorem, we get

$$F(t) = F(t_0) + h\,\dot{F}(t_0) + \frac{h^2}{\lfloor 2}\ddot{F}(t_0) + \frac{h^3}{\lfloor 3}\dddot{F}(t_0) + \ldots \quad \text{.....(1.3.6)}$$

Since t_0 is a solution of the equation (1.3.4) therefore $F(t_0) = 0$, then we have

$$F(t) = h\,\dot{F}(t_0) + \frac{h^2}{\lfloor 2}\ddot{F}(t_0) + \frac{h^3}{\lfloor 3}\dddot{F}(t_0) + \ldots \quad \text{.....(1.3.7)}$$

We have the following cases :

(i) If $\dot{F}(t_0) \neq 0$, then we say that the curve and the surface have a simple intersection at $\vec{r}\,(t_0)$.

(ii) If $\dot{F}(t_0)=0$, but $\ddot{F}(t_0)\neq 0$, then $F(t)$ is of second order of h and we say that t_0 is a double zero of $F(t)$ and in this case C and S have two points of contact (or contact of first order) at $\vec{r}\ (t_0)$.

(iii) If $\dot{F}(t_0)=0, \ddot{F}(t_0)=0$, but $\dddot{F}(t_0)\neq 0$, then $F(t)$ is of third order of h and we say that t_0 is a triple zero of $F(t)$ and in this case we say that C and S have three point contact or contact of second order.

(iv) In general if $\dot{F}(t_0)=0, \ddot{F}(t_0)=0, \ldots, F^{n-1}(t_0)=0$, but $f^{n}(t_0)\neq 0$, then $F(t)$ is of nth order of h and we say that C and S have a n point contact or contact of $(n-1)$th order.

INFLEXIONAL TANGENT

A straight line which meets the surface S in three coincident points *i.e.*, it has a second order point of contact is called inflexional tangent to the surface at that point.

Example: *Find the plane that has three point contact at the origin with the curve*

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1.$$

Sol. Let the equation of the plane at the origin be

$$l\,x + m\,y + n\,z = 0 \qquad .....(1)$$

The equations of the given curve are

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1 \qquad .....(2)$$

At origin,

$$t^4 - 1 = 0, \quad t^3 - 1 = 0, \quad t^2 - 1 = 0.$$

Clearly, $t = 1$ satisfies all of these three equations. Hence, at the origin, we have $t = 1$.

Now the points of intersection of the curve (2) and the surface (1) are given by the zeroes of the function

$$F(t) = l(t^4 - 1) + m(t^3 - 1) + n(t^2 - 1)$$

or $$F(t) = l t^4 + m t^3 + n t^2 - l - m - n \quad .....(3)$$

For three point contact, we should have

$$\dot{F}(t) = 0, \ddot{F}(t) = 0.$$

Now $$\dot{F} = 4l t^3 + 3m t^2 + 2nt = 0 \quad .....(4)$$

and $$\ddot{F} = 12l t^2 + 6m t + 2n = 0 \quad .....(5)$$

At the origin *i.e.* at $t = 1$, the equation (4) and (5) become

$$4l + 3m + 2n = 0,\ 12l + 6m + 2n = 0 \quad .....(6)$$

Solving equation (6), we get

$$\frac{l}{3} = \frac{m}{-8} = \frac{n}{6}$$

Hence the required equation of plane is

$$3x - 8y + 6z = 0.$$

OSCULATING PLANE

Definition : The osculating plane at a point P of a curve C of class greater then or equal to two is the limiting position of the plane passing through the tangent line at P and a neighbouring point Q on the curve C as $Q \to P$. (or which contains the tangent line at P and is parallel to the tangent at Q as $Q \to P$).

Alternative : Let P, Q, R be three points on a curve C, the limiting position of the plane PQR, when Q and R tend to P, is called the osculating plane at the point P.

EQUATION OF OSCULATING PLANE

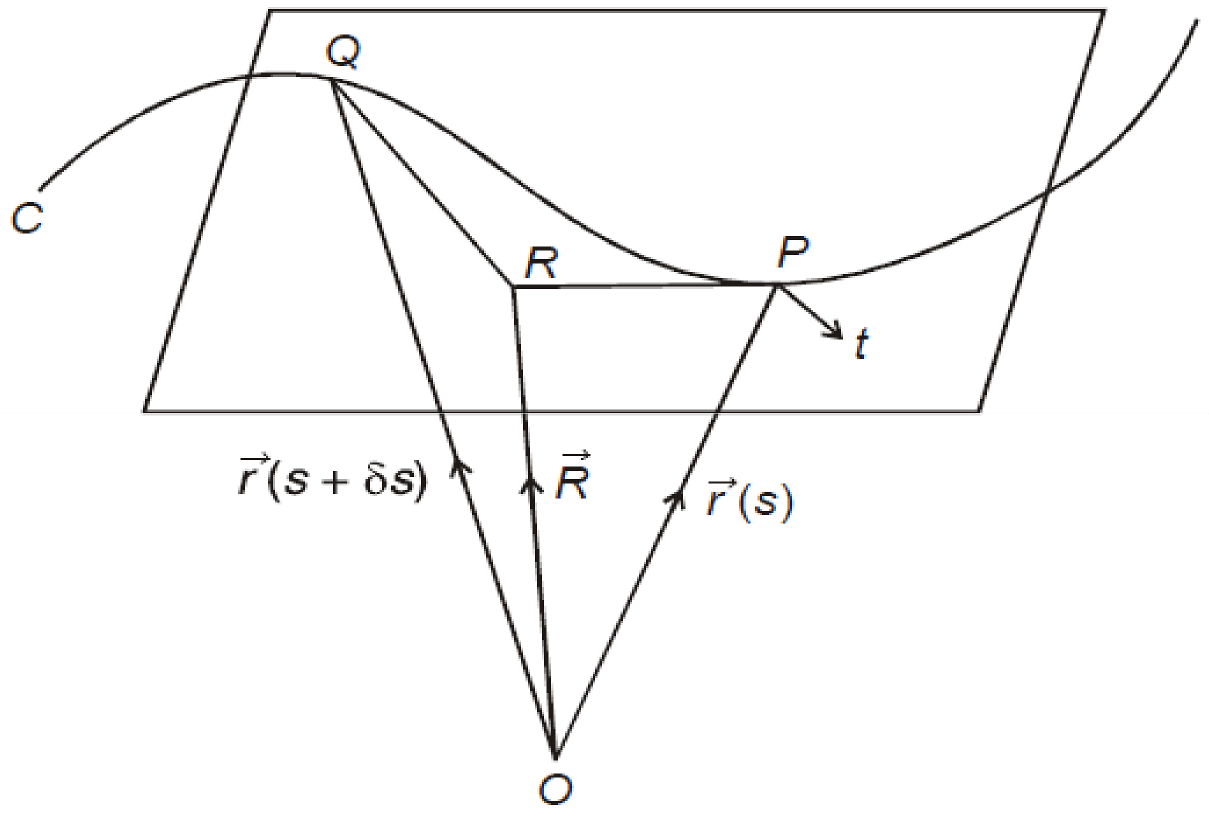

Fig. 1.3

Let $\vec{r} = \vec{r}(s)$ be the given curve C of class ≥ 2, in terms of parameter s, where s is the length of the arc of the curve measured from a fixed point on it. Let P and Q be two neighbouring points on the curve C with $\vec{r}(s)$ and $\vec{r}(s+\delta s)$ be their position vectors. Let $\vec{R}$ be the position vector of current point R on the plane containing the tangent line at P and the point Q.

Here $\qquad \overrightarrow{OP} = \vec{r}(s), \overrightarrow{OQ} = \vec{r}(s+\delta s), \overrightarrow{OR} = \vec{R}$

Hence $\qquad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r}(s+\delta s) - \vec{r}(s)$

and $\qquad \overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \vec{R} - \vec{r}(s)$

Again if $\hat{t}$ be the unit tangent vector at P,

then, $\qquad \hat{t} = \frac{d\vec{r}}{ds} = \vec{r}'(s).$

Now the vectors $\overrightarrow{PR}, \hat{t}$ and $\overrightarrow{PQ}$ are coplanar lying in the plane PQR and hence their scalar triple product is zero.

$$\left[\overrightarrow{PR},\ \hat{t},\ \overrightarrow{PQ}\right] = 0 \qquad \text{.....(1.4.1)}$$

or $$\left[\vec{R}-\vec{r}(s),\vec{r}'(s),\vec{r}(s+\delta s)-\vec{r}(s)\right]=0 \qquad .....(1.4.2)$$

but $$\vec{r}(s+\delta s)-\vec{r}(s)=\vec{r}'(s)\delta s+\vec{r}''(s)\frac{(\delta s)^2}{\underline{|2}}+..... \qquad .....(1.4.3)$$

We know that $[a\ b\ c] = a\cdot(b \times c)$.

Equation (1.4.2) may be written as

$$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\left[\vec{r}(s+\delta s)-\vec{r}(s)\right]=0 \qquad .....(1.4.4)$$

form (1.4.3) and (1.4.4)

$$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\left[\vec{r}'(s)\delta s+\vec{r}''(s)\frac{(\delta s)^2}{\underline{|2}}+...\right]=0 \qquad .....(1.4.5)$$

or $$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\left[\vec{r}''(s)\frac{(\delta s)^2}{\lfloor 2}+\text{ terms of higher order of }\delta s\right]=0$$

or $$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\left[\vec{r}''(s)+0\{\delta s\}\right]=0 \qquad .....(1.4.6)$$

The plane *PQR* tends to be the osculating plane when *Q* tends to *P* *i.e.* when $\delta s \to 0$, and hence the equation of the osculating plane is

$$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\vec{r}''(s)=0$$

or $$\left[\vec{R}-\vec{r}(s),\vec{r}'(s),\vec{r}''(s)\right]=0 \qquad .....(1.4.7)$$

Equation (1.4.7) represents the equation of the osculating plane in terms of parameter s of the point *P*.

Equation of the osculating plane in terms of general parameter t

Let $P(t)$ and $Q(t+\delta t)$ be the two neighbouring points on curve C. Let position vector of P and Q be $\bar{r}(t)$ and $\bar{r}(t+\delta t)$ with respect to origin, respectively.

The tangents at P and Q will be parallel to the vectors $\dot{\bar{r}}(t)$ and $\dot{\bar{r}}(t+\delta t)$, respectively.

Therefore the plane through the tangents at $P(t)$ and $Q(t+\delta t)$ is perpendicular to the vector

$$\dot{\bar{r}}(t)\times\dot{\bar{r}}(t+\delta t)$$

or to the vector $$\dot{\bar{r}}(t)\times\left[\dot{\bar{r}}(t+\delta t)-\dot{\bar{r}}'(t)\right] \quad \left[\because\ \dot{\bar{r}}(t)\times\dot{\bar{r}}(t)=0\right]$$

i.e. to the vector $$\dot{\bar{r}}(t)\times\frac{\dot{\bar{r}}(t+\delta t)-\dot{\bar{r}}'(t)}{\delta t} \qquad .....(1.4.8)$$

As $Q\to P$, $\delta t\to 0$ in this unit the osculating plane is perpendicular to the vector $\dot{\bar{r}}(t)\times\ddot{\bar{r}}(t)$.

If $\bar{R}$ be the position vector of any current point on the osculating plane, the equation of the osculating plane may be written as

$$\left(\bar{R}-\bar{r}\right)\cdot\dot{\bar{r}}\times\ddot{\bar{r}}=0 \quad \text{or} \quad \left[\bar{R}-\bar{r},\dot{\bar{r}},\ddot{\bar{r}}\right]=0 \qquad .....(1.4.9)$$

Equation of the osculating plane in terms of cartesian coordinates

Let the coordinates of a point P on a given curve C be (x, y, z) and coordinates of any current point be (X, Y, Z), these are functions of a parameter t.

Then $\qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $\qquad \vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$

Substituting these values in (1.4.9) the equation of the osculating plane is given by

$$\left[(X-x)\hat{i} + (Y-y)\hat{j} + (Z-z)\hat{k}, \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}, \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}\right] = 0$$

or
$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0 \qquad .....(1.4.10)$$

which is the equation of the osculating plane at a point $P(x, y, z)$.

Theorem : *To show that when the curve is analytic, there exists a definite osculating plane at a point of inflexion, provided the curve is not a straight line.*

Proof : We know that $\vec{r}'\left(=\hat{t}\right)$ is a unit tangent vector, therefore $\vec{r}'^2 = 1$. .....(1)

Differentiating w.r.t. 's' we get

$$2\vec{r}'\cdot\vec{r}'' = \vec{0} \quad \text{or} \quad \vec{r}'\cdot\vec{r}'' = \vec{0} \qquad .....(2)$$

Again differentiating, we get

$$\vec{r}''\cdot\vec{r}'' + \vec{r}'\cdot\vec{r}''' = \vec{0}$$

$$\vec{r}'\cdot\vec{r}''' = \vec{0} \qquad .....(3)$$

(At a point P where $\vec{r}'' = \vec{0}$, the tangent line is called inflexional and the point P is called the point of inflexion.)

If $\vec{r}''' \neq \vec{0}$, then r' is linearly independent of $\vec{r}'''$. Differentiating successively (3) and applying, above argument shall get

$$\vec{r}'\cdot\vec{r}^{m} = \vec{0}, \quad m \geq 2 \qquad .....(4)$$

where $\vec{r}^{m}$ is the first non-zero derivative of r at P.

Therefore if $\bar{r}^{m} \neq \bar{0}$, from equation (1.4.3), we get

$$\bar{r}(s+\delta s)-\bar{r}(s)=\frac{(\delta s)^{m}}{\underline{|m}}\cdot\bar{r}^{m}(s)+0\left\{(\delta s)^{m+1}\right\} \qquad .....(5)$$

Hence the equation of the osculating plane at P is

$$\left[\bar{R}-\bar{r}(s),\bar{r}'(s),\bar{r}^{m}(s)\right]=0 \qquad .....(6)$$

Again if for all $m \geq 2$ the derivative $\bar{r}^{m}=0$, we conclude $\bar{r}'\left(=\hat{t}\right)$ is constant (since the curve under consideration is analytic) *i.e.* the tangent vector is same at each point of the curve and hence the curve is a straight line.

Hence equation (6) is the equation of the osculating plane at a point of inflexion P when the curve is not straight line.

To find the osculating plane at a point of a space curve given by the intersection of two surfaces

Let the equations of the surfaces be

$$f(\bar{r}) = 0 \text{ and } g(\bar{r}) = 0 \qquad \text{.....(1.4.11)}$$

The equations of the tangent planes of these surfaces are given by

$$(\bar{R} - \bar{r}) \cdot \nabla f = 0 \text{ and } (\bar{R} - \bar{r}) \cdot \nabla g = 0 \qquad \text{.....(1.4.12)}$$

where ∇f and ∇g are normal vectors to $f(\bar{r}) = 0$ and $g(\bar{r}) = 0$ respectively and $\bar{R}$ be the position vector of current point on the plane.

The equation of the plane through the tangent line to the curve of intersection of the two surfaces is

$$F \equiv (\bar{R} - \bar{r}) \cdot \nabla f - \lambda (\bar{R} - \bar{r}) \cdot \nabla g = 0 \qquad \text{.....(1.4.13)}$$

If (1.4.13) be the equation of the osculating plane at P, it must have three point contact with the curve at P. Therefore the required conditions are

$$F = 0, \dot{F} = 0, \ddot{F} = 0 \, ; \qquad \text{.....(1.4.14)}$$

when $\bar{R} = \bar{r}$ and dashes denote differentiation with respect to parameter 't'.

$\dot{F} = 0$ gives

$$\dot{R} \cdot \nabla f + \left(\bar{R} - \bar{r}\right) \cdot \left(\nabla f\right)^{\square} - \lambda \, \dot{\bar{R}} \cdot \nabla g - \lambda \left(\bar{R} - \bar{r}\right) \cdot \left(\nabla g\right)^{\square} = 0 \qquad \text{.....(1.4.15)}$$

At $P, \bar{R} = \bar{r}$, condition (1.4.12) reduces to

$$\dot{\bar{r}} \cdot \nabla f - \lambda \, \dot{\bar{r}} \cdot \nabla g = 0 \qquad \text{.....(1.4.16)}$$

But we know that $\dot{\bar{r}}$ is a tangent vector and ∇f and ∇g are normal vectors to $f\left(\bar{r}\right) = 0$ and $g\left(\bar{r}\right) = 0$ and hence both

$$\dot{\bar{r}} \cdot \nabla f = 0 \text{ and } \dot{\bar{r}} \cdot \nabla g = 0 \qquad \text{.....(1.4.17)}$$

Hence $\dot{\bar{F}} = 0$ reduces to an identity.

Now consider the condition $\ddot{\bar{F}} = 0$ at $P, \bar{R} = \bar{r}$, we have

$$\ddot{\bar{r}} \cdot \nabla f = 0 - \lambda \, \ddot{\bar{r}} \cdot \nabla g = 0$$

or $$\lambda = \frac{\ddot{\vec{r}} \cdot \nabla f}{\ddot{\vec{r}} \cdot \nabla g} \qquad \text{.....(1.4.18)}$$

Now differentiating the equation (1.4.17), we get

$$\ddot{\vec{r}} \cdot \nabla f + \dot{\vec{r}} \cdot (\nabla f)^{\square} = 0, \; \ddot{\vec{r}} \cdot \nabla g + \dot{\vec{r}} \cdot (\nabla g)^{\square} = 0, \qquad \text{.....(1.4.19)}$$

or $$\ddot{\vec{r}} \cdot \nabla f = -\dot{\vec{r}} \cdot (\nabla f)^{\square}$$

and $$\ddot{\vec{r}} \cdot \nabla g = -\dot{\vec{r}} \cdot (\nabla g)^{\square}$$

$$\therefore \quad \frac{\ddot{\vec{r}} \cdot \nabla f}{\ddot{\vec{r}} \cdot \nabla g} = \frac{\dot{\vec{r}} \cdot (\nabla f)^{\square}}{\dot{\vec{r}} \cdot (\nabla g)^{\square}} = \lambda, \text{ from (1.4.18)} \qquad \text{.....(1.4.20)}$$

Putting the value of λ in (1.4.13), we get

$$\frac{(\vec{R}-\vec{r})\cdot\nabla f}{(\vec{R}-\vec{r})\cdot\nabla g}=\lambda=\frac{\dot{\vec{r}}\cdot(\nabla f)^{\square}}{\dot{\vec{r}}\cdot(\nabla g)^{\square}} \text{ form (1.4.20)}$$

or

$$\frac{(\vec{R}-\vec{r})\cdot\nabla f}{\dot{\vec{r}}\cdot(\nabla f)^{\square}}=\frac{(\vec{R}-\vec{r})\cdot\nabla g}{\dot{\vec{r}}\cdot(\nabla g)^{\square}} \qquad \text{.....(1.4.21)}$$

Above equation represents the equation of the osculating plane at P.

Cartesian form :

Let

$$f(\vec{r})=f(x,y,z),\ g(\vec{r})=g(x,y,z)$$

$$\vec{R}=Xi+Yj+Zk,\ \ \vec{r}=xi+yj+zk$$

$$\nabla f = \left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \left(\frac{\partial f}{\partial z}\right)\hat{k}$$

$$\nabla f = f_x\,\hat{i} + f_y\,\hat{j} + f_z\,\hat{k}$$

$$\Rightarrow \qquad \left(\nabla f\right)^{\square} = \Sigma\left(f_{xx}\,\dot{x} + f_{xy}\,\dot{y} + f_{xz}\,\dot{z}\right)\hat{i}$$

substituting in equation (1.4.21) of the osculating plane, we get

$$\frac{(X-x)f_x + (Y-y)f_y + (Z-z)f_z}{\left(\dot{x}^2 f_{xx} + \ldots + 2\dot{y}\dot{z}\, f_{yz} + \ldots\right)} = \frac{(X-x)g_x + (Y-y)g_y + (Z-z)g_z}{\left(\dot{x}^2 g_{xx} + \ldots + 2\dot{y}\dot{z}\, g_{yz} + \ldots\right)} \quad \ldots\ldots(1.4.22)$$

Example: *For the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$, show that any plane meets it in three points and deduce the equation to the osculating plane at $t = t_1$.*

Sol. Let the equation of the plane be

$$Ax + By + Cz + D = 0 \quad .....(1)$$

$$\therefore \quad F(t) = 3At + 3Bt^2 + 2Ct^3 + D = 0 \quad .....(2)$$

which is cubic in t. Hence the plane meets the given curve in three points.

Also $\quad \dot{x} = 3,\ \dot{y} = 6t,\ \dot{z} = 6t^2$

$$\ddot{x} = 0,\ \ddot{y} = 6,\ \ddot{z} = 12t \quad .....(3)$$

Hence the equation of osculating plane at the point t_1 is

$$\begin{vmatrix} x - 3t_1 & y - 3t_1^2 & z - 2t_1^3 \\ 3 & 6t_1 & 6t_1^2 \\ 0 & 6 & 12t_1 \end{vmatrix} = 0$$

or $\quad 2t_1^2 x - 2t_1 y + z = 2t_1^3 \quad$ is the required equation of the osculating plane at $t = t_1$.

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THANKS